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DECEMBER, 1859.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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MATHEMATICAL MONTHLY PRIZES.

We are informed that some of the conditions under which these Prizes are offered are not well understood; especially by those whose attention was first called to the matter in the last number of the Monthly. There are two classes of Prizes; I. For Solutions. In each number of the Monthly we give five problems, entitled *Prize Problems for Students*. The first and second of these problems are quite simple, and are intended for students in High Schools, Academies, and all Institutions not conferring degrees; and for the best solutions of which a *copy of the Monthly is given* as a prize. The third, fourth, and fifth problems are somewhat more difficult, and are open to the competition of all students, whether connected with an Educational Institution or not. For the solution of these problems, two prizes are offered; a first prize of *six dollars*, and a second prize of *four dollars*. Conditions: All the steps in each solution must be fully given; cuts, if any, must be neatly drawn, of the proper size, and the whole communicated in a plain and legible hand-writing. The name of the competitor, as well as the Institution with which he is connected, if any, must be written on a separate slip, which must also be signed by his Teacher, as evidence that the party is fairly entitled to compete for the prize. Credit will in all cases be given for all solutions received. II. For Essays: Five prizes are given for essays; a first prize of *fifty dollars*; a second prize of *forty dollars*; a third prize of *thirty dollars*; a fourth prize of *twenty dollars*; and a fifth prize of *ten dollars*. The first and second of these prizes are open to all competitors; the third, fourth, and fifth are open to all students, and under the same conditions as the prizes for the third, fourth, and fifth problems. . . . . Our contributors will much oblige us by using, as far as possible, the cuts already engraved for the Monthly. In many cases a few more lines and letters will adapt the cut to the new problem; and these additions can be readily made, thus saving us the trouble and expense of engraving an entirely new cut.

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TERMS.

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THE

MATHEMATICAL MONTHLY.

Vol. II. . . . DECEMBER, 1859. . . . No. III.

PRIZE PROBLEMS FOR STUDENTS.

I. If two circles touch each other, any straight line passing through the point of contact cuts off similar parts of their circumferences.

II. Find the four roots of the recurring equation

$$x^4 - \tfrac{5}{2}x^3 + 2x^2 - \tfrac{5}{2}x + 1 = 0.$$

III. If $2\cos\theta = u + \frac{1}{u}$, prove that $2\cos 2\theta = u^2 + \frac{1}{u^2}$, $2\cos 3\theta = u^3 + \frac{1}{u^3} \ldots\ldots 2\cos n\theta = u^n + \frac{1}{u^n}$; and then find the sum of the series, $\cos\theta + \cos 2\theta + \cos 3\theta \ldots\ldots + \cos n\theta$.

IV. Having given the Right Ascensions and Declinations of two stars, to find the formula for the distance between them. Also, find what the distance becomes, when for one star A. R. is $8^h\ 12^m\ 38^s.17$, and Dec. $17°\ 23'\ 49''.8$ north, and for the other A. R. is $13^h\ 28^m\ 19^s.92$, and Dec. $21°\ 12'\ 37''.2$ south.

V. In a frustum of any pyramid or cone, the area of a section, parallel to the two bases and equidistant from them, is the arithmetical mean of the arithmetical and geometrical means of the areas of the two bases.

The solutions of these problems must be received by February 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. XI. Vol. I.

THE first Prize is awarded to GUSTAVUS FRANKENSTEIN, Springfield, Ohio.

The second Prize is awarded to ASHER B. EVANS, Madison University, Hamilton, New York.

PRIZE SOLUTION OF PROBLEM I.

By GUSTAVUS FRANKENSTEIN, Springfield, Ohio.

Solve the equations

$$x+y=a$$
$$(x^3+y^3)(x^2+y^2)=b,$$

and give a discussion of the values of the roots.

Squaring and cubing $x+y=a$, we get

$$x^2+2xy+y^2=a^2,\ x^3+3x^2y+3xy^2+y^3=a^3.$$
$$\therefore x^2+y^2=a^2-2xy,\ x^3+y^3=a^3-3xy(x+y)=a^3-3axy.$$
$$\therefore (x^3+y^3)(x^2+y^2)=(a^3-3axy)(a^2-2xy)=b.$$
$$\therefore 6ax^2y^2-5a^3xy+a^5=b;$$

and solving we get

$$xy=\frac{5a^3\pm\sqrt{a(24b+a^5)}}{12a}=q.$$

Hence, knowing the sum, a, of x and y, and their product, q, their values will be given by the quadratic $x^2-ax+q=0$.
Solving we get

$$x=\frac{a}{2}\pm\tfrac{1}{2}\sqrt{a^2-4q}=\frac{a}{2}\pm\tfrac{1}{2}\sqrt{-\tfrac{1}{6}a^2\mp\frac{\sqrt{a(24b+a^5)}}{12a}}.$$

DISCUSSION. — *Case* I. *When a and b have the same signs.* Since $-\frac{1}{6}a^2$ is negative, whether a be positive or negative, it is evident that two of these values of x will always be imaginary. If a and b are of the same sign, q is real, since $a(24b+a^5)$ is positive; and x

will have two real values when $\sqrt{a\,(24\,b + a^5)} > 2\,a^3$; or when $a\,(24\,b + a^5) > 4\,a^6$; or $8\,b > a^5$. If this condition be not fulfilled, all the roots are imaginary. When $8\,b = a^5$ the roots are equal.

Case II. *When a and b have different signs.* In this case $\sqrt{a\,(24\,b + a^5)}$ will be imaginary unless $24\,b < a^5$, after which, that x may be real, $8\,b$ should be $=$ or $> a^5$; but as these conditions are contradictory, all the roots in this case are imaginary. Hence the conditions that there may be two real roots are, first, that a and b must have the same sign; and second, that $8\,b =$ or $> a^5$ numerically.

Indeed, by referring to the given equations, we see that if x and y are real, $x^2 + y^2$ is positive; and since $x^3 + y^3$ is always of the same sign as $x + y = a$, therefore b will also be of the same sign as a. If we suppose the roots equal, the proposed equations become $2\,x = a$, and $4\,x^5 = b$. Therefore, $4\left(\frac{a}{2}\right)^5 = b$; or $8\,b = a^5$; which, as before, exhibits the transition from real to imaginary roots.

If $\frac{1}{2}\,a + z$ and $\frac{1}{2}\,a - z$ be substituted for x and y in the proposed equations, the resulting equation in z will be

$$48\,a\,z^4 + 16\,a^3\,z^2 + a^5 - 8\,b = 0.$$

Now a being real, z must be real, for all real values of $x = \frac{1}{2}\,a \pm z$; therefore, for real values of x, z^4 and z^2 are essentially positive. But in this equation the term containing z^3 is wanting, and since the signs of its adjacent terms are the same, whether a be positive or negative, it appears from DESCARTES's theory of signs that z has always two imaginary roots; and when the signs of a and b are unlike, all the terms will be either positive or negative for real values of z, and therefore their sum cannot be nothing; hence, $x = \frac{1}{2}\,a \pm z$ cannot be real when a and b have different signs. But when a and b have like signs, there will be two real roots when $8\,b > a^5$, for then the equation, having its last term negative, cannot have all its roots imaginary. When $8\,b = a^5$, $z^2 = 0$, which corre-

sponds to the equal roots of $x = \frac{1}{2}a \pm 0 = y$, the other factor giving the imaginary roots.

Prize Solution of Problem II.

By Gustavus Frankenstein, Springfield, Ohio.

Let A, B, C be the angles, and a, b, c the opposite sides, of a plane triangle; it is required from the relation

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

to deduce the formula

$$a^2 = b^2 + c^2 - 2\,b\,c \cos A.$$

Since $A + B + C = 180°$, $C = 180° - (A + B)$; and

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

gives

$$\begin{aligned} (1.) \qquad c \sin A &= a \sin C = a \sin(A + B) \\ &= a\,(\sin A \cos B + \cos A \sin B). \end{aligned}$$

Substitute for $\sin B$ its value $\frac{b \sin A}{a}$, and for $\cos B$ its value $\frac{\sqrt{a^2 - b^2 \sin^2 A}}{a}$, and (1) becomes, after dividing by $\sin A$,

$$c = \sqrt{a^2 - b^2 \sin^2 A} + b \cos A.$$

$$\therefore\, a^2 - b^2 \sin^2 A = (c - b \cos A)^2 = c^2 - 2\,b\,c \cos A + b^2 \cos^2 A;$$

$$\begin{aligned} \therefore\, a^2 &= c^2 + b^2 (\sin^2 A + \cos^2 A) - 2\,b\,c \cos A \\ &= c^2 + b^2 - 2\,b\,c \cos A. \end{aligned}$$

Prize Solution of Problem III.

By Asaph Hall, Assistant at Harvard College Observatory.

A number n of equal circles touch each other externally, and include an area of a square feet; to find the radii of the circles. — Communicated by Artemas Martin, Esq.

Joining the centres of the equal circles, we shall have a regular polygon of n sides, each side being equal to $2\,r$, twice the required radius.

Join the centre of the polygon with the centre of each circle. In each of the n equal triangles thus formed, the angle at the centre of the polygon is one nth of four right angles, or $\frac{2\pi}{n}$; and the sum of the remaining angles of each triangle is $\pi - \frac{2\pi}{n} = \frac{(n-2)\pi}{n} =$ the angle of each of the n equal sectors. The altitude of each triangle is $r \cot \frac{\pi}{n}$, and therefore $r^2 \cot \frac{\pi}{n}$ is the area of each triangle, and $n r^2 \cot \frac{\pi}{n}$ the area of the polygon. The area of each sector is $\frac{r^2 (n-2)\pi}{2n}$, and of the n sectors, $\frac{r^2(n-2)\pi}{2}$.

$$\therefore n r^2 \cot \frac{\pi}{n} - \tfrac{1}{2} r^2 (n-2)\pi = a,$$

$$\therefore r = \sqrt{\frac{2a}{2n \cot\frac{\pi}{n} - (n-2)\pi}}.$$

Prize Solution of Problem IV.

By David Trowbridge, Perry City, Schuyler Co., N. Y.

If the sides of a spherical trapezium be denoted by a, b, c, d, the diagonals by δ_1 and δ_2, and the distance between the middle points of the diagonals by $\varDelta$; show that

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \tfrac{1}{2}\delta_1 \cos \tfrac{1}{2}\delta_2 \cos \varDelta.$$

— Communicated by George Eastwood, Esq.

Let B and P be the middle points of the diagonals δ_1 and δ_2, then in the triangles FBH, HBD, DBI, IBF, we have from a fundamental formula of spherical trigonometry

$$\cos b = \cos \tfrac{1}{2}\delta_1 \cos BH + \sin \tfrac{1}{2}\delta_1 \sin BH \cos HBF,$$
$$\cos a = \cos \tfrac{1}{2}\delta_1 \cos BH + \sin \tfrac{1}{2}\delta_1 \sin BH \cos HBD,$$
$$\cos c = \cos \tfrac{1}{2}\delta_1 \cos BI + \sin \tfrac{1}{2}\delta_1 \sin BI \cos DBI,$$
$$\cos d = \cos \tfrac{1}{2}\delta_1 \cos BI + \sin \tfrac{1}{2}\delta_1 \sin BI \cos IBF,$$
$$\therefore \cos a + \cos b + \cos c + \cos d = 2\cos \tfrac{1}{2}\delta_1 (\cos BH + \cos BI),$$

since HBF and HBD, as well as DBI and IBF, are supplementary, their cosines having opposite signs. But in the triangles BIP and BHP we have

$$\cos BH = \cos \tfrac{1}{2}\delta_2 \cos \Delta + \sin \tfrac{1}{2}\delta_2 \sin \Delta \cos BPH,$$
$$\cos BI = \cos \tfrac{1}{2}\delta_2 \cos \Delta + \sin \tfrac{1}{2}\delta_2 \sin \Delta \cos BPI,$$

$\therefore \cos BH + \cos BI = 2 \cos \tfrac{1}{2}\delta_2 \cos \Delta$, since the angles BPH and BPI are supplementary.

$$\therefore \cos a + \cos b + \cos c + \cos d = 2 \cos \tfrac{1}{2}\delta_1 (\cos BH + \cos BI)$$
$$= 4 \cos \tfrac{1}{2}\delta_1 \cos \tfrac{1}{2}\delta_2 \cos \Delta.$$

Prize Solution of Problem V.

By Asher B. Evans, Madison University, Hamilton, N. Y.

From an urn containing four white and four black balls, four are repeatedly drawn and replaced. A agrees to pay B one dollar every time the four balls drawn are equally divided between white and black; but if three, or all four, are of the same color, B is to pay A one dollar. Who has the advantage, and what is its value for each drawing? — Communicated by Simon Newcomb, Esq.

The whole number of different fours which can be drawn out of the eight balls is

$$\overset{4}{\underset{8}{C}} = \frac{8\,(8-1)\,(8-2)\,(8-3)}{1\,.\,2\,.\,3\,.\,4} = 70.$$

Since "A agrees to pay B one dollar every time the four balls are equally divided between white and black," the number of the 70 in B's favor is

$$\overset{2}{\underset{4}{C}} \times \overset{2}{\underset{4}{C}} = 36,$$

and the number in A's favor, since three of the four may be white and one black, three black and one white, all four white, all four black, is

$$2\,(\overset{3}{\underset{4}{C}} \times \overset{1}{\underset{4}{C}} + \overset{4}{\underset{4}{C}}) = 34.$$

Therefore B has the advantage, which amounts to two dollars in 70 draws, or one thirty-fifth of a dollar for each drawing.

Chauncey Wright.
Truman Henry Safford.

NOTES AND QUERIES.

1. *The greatest Common Divisor.* *Case* I. *When the given numbers are such as may be readily factored.* The following process is based upon the well-known principle of dividing by component factors. It is evident that if several numbers have a common divisor, they may all be divided by any component factor of this divisor, and the resulting quotients by another of the component factors; and so on.

EXAMPLE.

2	84	126	210	252	294	462
3	42	63	105	126	147	231
7	14	21	35	42	49	77
	2	3	5	6	7	11

Greatest common divisor $= 2 \times 3 \times 7 = 42$.

Case II. *When the factors of the given numbers are not readily apparent.* The well-known rule of dividing the greater by the less, the last divisor by the last remainder, &c., need not be repeated. But the *method*, or *form of work*, given here, is recommended to teachers as being more concise and elegant than the usual method, requiring less time and less space on the slate or blackboard.

OPERATION.

		167463
73761	2	147522
59823	3	19941
13938	1	13938
12006	2	6003
1932	3	5796
1863	9	207
69	3	207

Ans. 69.

EXAMPLE.

What is the greatest common divisor of 73761 and 167463?

The operation needs no explanation. —J. C. PORTER, Professor of Mathematics in Clinton Liberal Institute, Clinton, N. Y.

2. *Equation of Payments.*—There is probably no mercantile calculation that is more tedious than the averaging of accounts, or Equation of Payments. The labor of computation may be much

diminished by performing all the multiplications at once,— first multiplying by all the units' figures of the several multipliers, then by all the tens' figures, and so on.

EXAMPLE.

Required the average time of payment of $ 371 due in 15 days, $ 25 due in 84 days, $ 1603 due in 107 days, and $ 885 due in 138 days.

$$\begin{array}{r}
371 \times 15 \\
25 \times 84 \\
1603 \times 107 \\
885 \times 138 \\
\hline
20256 \\
3226 \\
2488 \\
\hline
301316
\end{array}$$

Commencing with the units' figures of the multipliers, say $8 \times 5 + 7 \times 3 + 4 \times 5 + 5 \times 1 = 86$. Set down 6, and carry 8. $8 + 8 \times 8 + 7 \times 0 + 4 \times 2 + 5 \times 7 = 115$. Set down 5, and carry 11. $11 + 8 \times 8 + 7 \times 6 + 5 \times 3 = 132$. Set down 2, and carry 13. $13 + 7 \times 1 = 20$, which is to be set down.

$3 \times 5 + 0 \times 3 + 8 \times 5 + 1 \times 1 = 56$. Set down 6 in tens' place, and carry 5. $5 + 3 \times 8 + 0 \times 0 + 8 \times 2 + 1 \times 7 = 52$. Set down 2, and carry 5. $5 + 3 \times 8 + 0 \times 6 + 1 \times 3 = 32$, which set down.

$1 \times 5 + 1 \times 3 = 8$, which is to be set down in hundreds' place. $1 \times 8 + 1 \times 0 = 8$, which set down. $1 \times 8 + 1 \times 6 = 14$. Set down 4, and carry 1. $1 + 1 \times 1 = 2$, which set down. Adding these partial products, we find the sum of the several products is 301316, as may be easily proved by multiplying in the usual way.

A little practice will enable one to perform the multiplications with nearly as much facility as simple addition. The *products* should be mentally announced and added together, thus: 40 and 21 and 20 and 5 are 86. 8 and 64 and 8 and 35 are 115. 11 and 64 and 42 and 15 are 132. 13 and 7 are 20, &c., &c.— PLINY EARLE CHASE, Philadelphia, Pa.

3. *Decomposition of the irreducible rational fraction* $\frac{f(x)}{((x-\alpha)+\beta^2)^n}$ *into simple fractions of the forms*

$$\frac{ax+b}{((x-\alpha)^2+\beta^2)^n},\ \frac{a'x+b'}{((x-\alpha)^2+\beta^2)^{n-1}},\ \&c.$$

Divide $f(x)$ by $(x-\alpha)^2+\beta^2$; the remainder of this division will be the numerator of the first fraction sought. For the division will give an equality of the form

$$f(x)=((x-\alpha)^2+\beta^2)\,f_1(x)+a\,x+b\,;$$

whence

$$\frac{f(x)}{((x-\alpha)^2+\beta^2)^n}=\frac{ax+b}{((x-\alpha)^2+\beta^2)^n}+\frac{f_1(x)}{((x-\alpha)^2+\beta^2)^{n-1}}.$$

By dividing in like manner the quotient $f_1(x)$ by $(x-\alpha)^2+\beta^2$, the remainder of this second division will be the numerator $a'x+b'$ of the second fraction sought; and so on. — *Nouvelles Annales de Mathématiques*, Septembre, 1859.

The student will see the simplicity of this method of decomposition from the following example. Decompose $\frac{5x^3+6x^2-8x+20}{(x-2)^4}$.

$$\begin{aligned}\frac{5x^3+6x^2-8x+20}{(x-2)^4}&=\frac{68}{(x-2)^4}+\frac{5x^2+16x+24}{(x-2)^3}\\&=\frac{68}{(x-2)^4}+\frac{76}{(x-2)^3}+\frac{5x+26}{(x-2)^2}\\&=\frac{68}{(x-2)^4}+\frac{76}{(x-2)^3}+\frac{36}{(x-2)^2}+\frac{5}{x-2}.\end{aligned}$$

It is obvious that this method can be applied to any fraction of the form $\frac{f(x)}{(F(x))^n}$, in which $f(x)$ is of a higher degree than $F(x)$.

REVIEW OF THE PRIZE SOLUTION OF THE LAST PROBLEM IN EMERSON'S NORTH AMERICAN ARITHMETIC.*

By Hon. Finley Bigger, Register U. S. Treasury, Washington, D. C.

The following review of the Prize Solution of Problem 137 in Emerson's North American Arithmetic, Part III., was submitted to the National Teachers' Association at its late session in Washington, and referred to the Mathematical Monthly for publication.

For the purpose of elucidation, it is assumed that the question is susceptible of two constructions. The one adopted in the Prize Solution considers each term of supposition, compared with the term of demand, as separate and distinct propositions, and that the words "the grass being at first equal on every acre, and growing uniformly," demand that the acres, in each condition of the question speci-

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* Prize Question 137. — "If 12 oxen eat up $3\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks; the grass being, at first, equal on every acre, and growing uniformly?"

In June, 1835, a premium of $ 50 was offered for the most "lucid analytical solution" of the last question in the Third Part of Emerson's North American Arithmetic; and subsequently a committee to examine the solutions presented, and award the premium, was raised in the manner proposed. The committee have given a very careful and patient attention to the labors of the trust confided to them, and they now make the following

REPORT.

The whole number of solutions presented was 112, of which 48 gave the true answer. After excluding those solutions which gave incorrect answers, the committee proceeded to diminish the remaining number, by excluding those which were algebraical, and also those which were performed either by *position* or *proportion*, retaining for the comparative examination such only as were strictly analytical. The solution for which the committee have awarded the premium was presented by James Robinson, Principal of the Department of Arithmetic, Bowdoin School, Boston. It is as follows: —

Solution. — It is evident that a part of the given number of oxen, in each condition of this question, must be supported by the grass *at first standing* on the given number of acres, and that the remaining part must be supported by the *growth*. It is also evident that the number of oxen that can be supported by the grass at first standing on the ground must be in a direct ratio to the number of acres, and in an inverse ratio to the time of grazing. And it is further obvious that the number of oxen that can be supported by the growth of the grass must be in a direct ratio to the number of acres, without any regard to the *time* of grazing, because the

fied, shall be so increased proportionally, as that the answer to each supposition, thus separately considered, shall be precisely the same in amount, or such that one answer will be alike the ratio of either.

The other insists that the words "*being* equal and *growing* uniformly," are merely suggestive of condition, and do not authorize an increase of the numerical expression of the acres of grass specified in the terms of the proposition.

This second interpretation regards the alleged qualifying language as indicating no mathematical ratio, or measure of value, and concludes, therefore, that the solution of the question must proceed as if this phraseology were omitted, and the acres designated as tons of hay. And, thus considered, but one answer is possible, and this the expression of the mean ratio of the two terms of supposition through the one term of demand.

We offer a few words by way of analysis and criticism of the

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number of oxen that would consume the growth of any given number of acres during any given time, would consume the same growth continually.

By the first condition of the question, 12 oxen consume $3\frac{1}{2}$ acres of grass and its growth in 4 weeks; the 10 acres being $\frac{20}{7}$ of $3\frac{1}{2}$ acres, it would require $\frac{20}{7}$ as many oxen to consume 10 acres of grass and its growth in the same time; and 12 oxen multiplied by $\frac{20}{7}$ are $34\frac{2}{7}$ oxen. To consume the same in 9 weeks would require only $\frac{4}{9}$ as many oxen, and $34\frac{2}{7}$ oxen multiplied by $\frac{4}{9}$ are $15\frac{5}{21}$ oxen.

By the second condition, 21 oxen consume 10 acres of grass and its growth in 9 weeks, and 21 oxen less $15\frac{5}{21}$ oxen are $5\frac{16}{21}$ oxen. Then it follows, that $5\frac{16}{21}$ oxen in 9 weeks would consume the growth of 10 acres of grass during the 5 remaining weeks. To consume the growth of 10 acres during 9 weeks would require $\frac{9}{5}$ as many oxen, and $5\frac{16}{21}$ oxen multiplied by $\frac{9}{5}$ are $10\frac{13}{35}$ oxen. Then, 21 oxen less $10\frac{13}{35}$ oxen are $10\frac{22}{35}$ oxen. Hence it is evident that $10\frac{22}{35}$ oxen, in 9 weeks, would consume the grass at first on the 10 acres; and it is also evident that $10\frac{13}{35}$ oxen, in 9 weeks, would consume the growth of the 10 acres of grass during the 9 weeks.

The 24 acres in the third condition being $\frac{24}{10}$, or $2\frac{2}{5}$ times 10 acres, it would require $2\frac{2}{5}$ times $10\frac{22}{35}$ oxen to consume the grass at first on the 24 acres in 9 weeks; and $10\frac{22}{35}$ oxen multiplied by $2\frac{2}{5}$ are $25\frac{89}{175}$ oxen. To consume the same in 18 weeks would require only $\frac{9}{18}$, or $\frac{1}{2}$ as many oxen; and $25\frac{89}{175}$ oxen, divided by 2, are $12\frac{132}{175}$ oxen. And to consume the growth of the 24 acres of grass during the 18 weeks would require $2\frac{2}{5}$ times $10\frac{13}{35}$ oxen; and $10\frac{13}{35}$ oxen multiplied by $2\frac{2}{5}$ are $24\frac{156}{175}$ oxen.

Lastly, $12\frac{132}{175}$ oxen plus $24\frac{156}{175}$ oxen are $37\frac{113}{175}$ oxen, the number required.

By order of the Committee,

P. MACKINTOSH, *Chairman.*

Prize Solution. By analysis the author arrived at the following results. "Hence, it is evident that $10\frac{22}{35}$ oxen, in 9 weeks, would consume the grass at first standing on the 10 acres; and it is also evident that $10\frac{13}{35}$ oxen, in 9 weeks, would consume the growth of 10 acres during the 9 weeks." If 10 acres, in 9 weeks, grow 10 acres, 24 acres in 18 weeks would grow 48 acres; and thus the Prize Solution presents us with the following propositions, which produce the final result. If $10\frac{22}{35}$ oxen, in 9 weeks, eat the grass at first on 10 acres, how many oxen will eat the grass at first on the 24 acres, in 18 weeks? and if $10\frac{13}{35}$ oxen, in 9 weeks, eat the growth of 10 acres, how many oxen will eat the growth of 24 acres, which is 48, in 18 weeks? $\frac{24 \times 9 \times 10\frac{22}{35}}{10 \times 18} = 12\frac{132}{175}$ oxen; $\frac{48 \times 9 \times 10\frac{13}{35}}{10 \times 18} = 24\frac{156}{175}$ oxen, and $12\frac{132}{175} + 24\frac{156}{175} = 37\frac{113}{175}$ oxen, the same as given by the Prize Solution. If this be what it purports, the true answer, by inverting the oxen in the above statement, the same answer *must* be obtained.

Thus $\frac{24 \times 9 \times 10\frac{13}{35}}{10 \times 18} = 12\frac{78}{175}$ oxen; $\frac{48 \times 9 \times 10\frac{22}{35}}{10 \times 18} = 25\frac{89}{175}$ oxen, and $12\frac{78}{175} + 25\frac{89}{175} = 37\frac{167}{175}$ oxen, the answer. And thus is exhibited the fact that this solution, as well as $37\frac{113}{175}$, fulfils the conditions of the question; for $37\frac{167}{175}$ oxen is equally the answer with $37\frac{113}{175}$ oxen.

The second interpretation furnishes the following solution. If 12 oxen, in 4 weeks, eat $3\frac{1}{2}$ acres of grass, how many oxen will eat 24 acres in 18 weeks? And, if 21 oxen, in 9 weeks, eat 10 acres of grass, how many oxen will eat 24 acres in 18 weeks? First $\frac{24 \times 12 \times 4}{3\frac{1}{2} \times 18} = 18\frac{2}{7}$ oxen; second, $\frac{24 \times 21 \times 9}{10 \times 18} = 25\frac{1}{5}$ oxen, and $18\frac{2}{7} + 25\frac{1}{5} = 43\frac{17}{35}$ oxen; $24 + 24 = 48$ acres. If then, $43\frac{17}{35}$ oxen, in 18 weeks, eat 48 acres of grass, it is self-evident that it will require $\frac{1}{2}$ of $43\frac{17}{35}$ oxen to eat $\frac{1}{2}$ the number of acres, 24, in

the same length of time, 18 weeks, and $\frac{1}{2}$ of $43\frac{17}{35} = 21\frac{26}{35}$ oxen, the answer.

Note. — We have appended the Prize Solution, to give our readers an opportunity to judge for themselves. It does not seem to us that there is any ambiguity in the statement of the problem, nor much difficulty in its solution; and we add the following for those who may wish to study it. Let *unity* denote the amount of grass at first on each acre, g the growth on each acre per week, and x the required number of oxen. Then, from the first condition, $1 + 4g$ is the amount of grass consumed from one acre in 4 weeks, $(1 + 4g)\,3\frac{1}{3}$ the amount from $3\frac{1}{3}$ acres, $\frac{(1+4g)\,3\frac{1}{3}}{12}$ the amount consumed by one ox in 4 weeks, $\frac{(1+4g)\,3\frac{1}{3}}{12 \times 4}$ the amount consumed by one ox in one week. From the second condition, $1 + 9g$ is the amount of grass consumed from one acre in 9 weeks, $(1 + 9g)\,10$ the amount from 10 acres, $\frac{(1+9g)\,10}{21}$ the amount consumed by one ox in 9 weeks, $\frac{(1+9g)\,10}{21 \times 9}$ the amount consumed by one ox in 1 week. From the third condition, $\frac{(1+18g)\,24}{x \times 18}$ is the amount consumed by one ox in 1 week. Therefore

$$\frac{(1+4g)\,3\frac{1}{3}}{12 \times 4} = \frac{(1+9g)\,10}{21 \times 9}$$

$$\frac{(1+18g)\,24}{x \times 18} = \frac{(1+9g)\,10}{21 \times 9}.$$

By solving these equations we get $x = 37\frac{11}{178}$, which is the only number of oxen which will satisfy all the conditions of the problem.

SOLUTION OF CUBIC EQUATIONS BY THE COMMON LOGARITHMIC TABLES.*

By a Correspondent.

By removing the second term, in the usual mode, every cubic takes the form $x^3 + ax = b$. Assume $x = y + z$, and $3yz = -a$. By substituting these, the given equation will become $y^3 + z^3 = b$. From the last two equations, the value of y^3 and of z^3 can be found by a quadratic. Since $x = y + z$, we thus obtain Cardan's formula,

* For the trigonometrical solution of equations of the second, third, and fourth degrees, the student may consult Cagnoli's *Trigonométrie*, Chap. XIV.; or Chauvenet's *Trigonometry*, pp. 95 - 100, for the solution of equations of the second and third degrees. — Ed.

$$x = \left(\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} + \left(\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}, \text{ or}$$

$$= \left(\frac{b}{2}\right)^{\frac{1}{3}} \left\{ \left(1 + \sqrt{1 + \left(\frac{2}{b}\right)^2 \left(\frac{a}{3}\right)^3}\right)^{\frac{1}{3}} + \left(1 - \sqrt{\left(\frac{2}{b}\right)^2 \left(\frac{a}{3}\right)^3}\right)^{\frac{1}{3}} \right\}.$$

In adapting this expression to the logarithmic tables three cases are presented.

I. *When a is positive.* Make $\tan v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$, or $\frac{b}{2\cos v} = \left(\frac{a}{3}\right)^{\frac{3}{2}} \div \sin v$; then by trigonometry,

$$x = \left(\frac{b}{2}\right)^{\frac{1}{3}} \{(1 + \sqrt{1 + \tan^2 v})^{\frac{1}{3}} + (1 - \sqrt{1 + \tan^2 v})^{\frac{1}{3}}\},$$

$$= \left(\frac{b}{2}\right)^{\frac{1}{3}} \left\{ \left(1 + \frac{1}{\cos v}\right)^{\frac{1}{3}} + \left(1 - \frac{1}{\cos v}\right)^{\frac{1}{3}} \right\},$$

$$= \sqrt{\frac{a}{3}} \left\{ \left(\frac{1 + \cos v}{\sin v}\right)^{\frac{1}{3}} - \left(\frac{1 - \cos v}{\sin v}\right)^{\frac{1}{3}} \right\},$$

$$= \sqrt{\frac{a}{3}} \left\{ \left(\cot \frac{v}{2}\right)^{\frac{1}{3}} - \left(\tan \frac{v}{2}\right)^{\frac{1}{3}} \right\}.$$

This equation may be further simplified by making $\tan u = \left(\tan \frac{v}{2}\right)^{\frac{1}{3}}$; then

$$x = \sqrt{\frac{a}{3}} (\cot u - \tan u).$$

$$= 2\sqrt{\frac{a}{3}} \cot 2u.$$

In this case there is but one real root. When b is negative it will only change the signs of v and u.

II. *When a is negative and of such value that $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ is less than* 1.

Let $\sin v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$; then

$$x = \left(\frac{b}{2}\right)^{\frac{1}{3}} \{(1 + \cos v)^{\frac{1}{3}} + (1 - \cos v)^{\frac{1}{3}}\}$$

$$= (b)^{\frac{1}{3}} \left\{ \left(\cos \frac{v}{2}\right)^{\frac{2}{3}} + \left(\sin \frac{v}{2}\right)^{\frac{2}{3}} \right\}.$$

For logarithmic computation, let

$$\tan u = \left(\tan\frac{v}{2}\right)^{\frac{1}{3}},\ \text{or}\ \frac{\cos^{\frac{2}{3}}\frac{v}{2}}{\cos^2 u} = \frac{\sin^{\frac{2}{3}}\frac{v}{2}}{\sin^2 u} = \frac{\left(\cos\frac{v}{2}\sin\frac{v}{2}\right)^{\frac{1}{3}}}{\cos u \sin u}.$$

$$x = b^{\frac{1}{3}} \cos^{\frac{2}{3}}\frac{v}{2}\,(1 + \tan^2 u) = \frac{b^{\frac{1}{3}}\cos^{\frac{2}{3}}\frac{v}{2}}{\cos^2 u}$$

$$= \frac{b^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{3}}\sin^{\frac{1}{3}} v}{\frac{1}{2}\sin 2u} = \frac{2\sqrt{\frac{a}{3}}}{\sin 2u}.$$

In this case, also, there is but one real root; and when b is negative, the arcs v and u will be negative, as before.

III. " *The irreducible case.*" *When a is negative and of such value that $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ is greater than* 1.

Make $\cos v = \frac{b}{2}\left(\frac{3}{a}\right)^{\frac{3}{2}}$; then

$$x = \sqrt{\frac{a}{3}}\,\{(\cos v + \sqrt{-1}\sin v)^{\frac{1}{3}} + (\cos v - \sqrt{-1}\sin v)^{\frac{1}{3}}\}.$$

By applying Demoivre's theorem, the imaginary quantities disappear, leaving $x = 2\sqrt{\frac{a}{3}}\cos\frac{v}{3}$. But this $\cos\frac{v}{3}$ corresponds to the arcs v, $360° + v$, and $360° - v$. Dividing by 3, and putting 120° under the form of 180° — 60°, we find the other two roots,

$$x = -2\sqrt{\frac{a}{3}}\cos\left(60° - \frac{v}{3}\right),$$

$$x = -2\sqrt{\frac{a}{3}}\cos\left(60° + \frac{v}{3}\right).$$

We are now prepared to recapitulate; it being recollected that 10 should be algebraically subtracted from the index of the logarithmic sines and tangents. First bring the given cubic to the general form $x^3 + a\,x = b$.

I. When the coefficient a is positive. Find

$$\tan v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}};\ \tan u = \left(\tan\frac{v}{2}\right)^{\frac{1}{3}};\ x = 2\sqrt{\frac{a}{3}}\cot 2u.$$

II. When a is negative, and $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ less than 1. Find

$$\sin v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}; \ \tan u = \left(\tan\frac{v}{2}\right)^{\frac{1}{3}}; \ x = \frac{2\sqrt{\frac{a}{3}}}{\sin 2u}.$$

III. When a is negative, and $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ greater than 1. Find

$$\cos v = \frac{b}{2}\left(\frac{3}{a}\right)^{\frac{3}{2}}; \ x = 2\sqrt{\frac{a}{3}}\cos\frac{v}{3}; \ x = -2\sqrt{\frac{a}{3}}\cos\left(60^\circ - \frac{v}{3}\right); \text{ and}$$

$$x = -2\sqrt{\frac{a}{3}}\cos\left(60^\circ + \frac{v}{2}\right).$$

In the case where $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}} = 1$, $x = 2\left(\frac{b}{2}\right)^{\frac{1}{3}}$.

Thus all the real roots of any cubic equation may be found by logarithms. It is perhaps unnecessary to remark, that in these values of x, the coefficient a is to be taken as numerically positive, irrespective of its algebraic sign. The investigation of these solutions is new in part, and will be found convenient for reference.

EQUATIONS OF THE SECOND DEGREE.

(1.) $x^2 + px = q$.

SOLUTION.

$$\tan A = \frac{2}{p}\sqrt{q},$$
$$x = \sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = -\sqrt{q}\cot\tfrac{1}{2}A,$$

(2.) $x^2 - px = q$.

SOLUTION.

$$\tan A = \frac{2}{p}\sqrt{q},$$
$$x = -\sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = \sqrt{q}\cot\tfrac{1}{2}A.$$

(3.) $x^2 + px = -q$.

(4.) $x^2 - px = -q$.

If $p^2 < 4q$ the roots of (3) and (4) are imaginary.

SOLUTION.

$$\sin A = \frac{2}{p}\sqrt{q},$$
$$x = -\sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = -\sqrt{q}\cot\tfrac{1}{2}A.$$

SOLUTION.

$$\sin A = \frac{2}{p}\sqrt{q},$$
$$x = \sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = \sqrt{q}\cot\tfrac{1}{2}A.$$

We have taken the liberty to add trigonometrical solutions of equations of the second degree, and commend them, as well as Correspondent's cubics, to the attention of students. — ED.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 406, Vol. I.]

SECTION V.

ON THE MOTIONS OF THE ATMOSPHERE ARISING FROM LOCAL DISTURBANCES.

58. BESIDES the general disturbance of equilibrium arising from a difference of specific gravity between the equator and the poles, which causes the general motions of the atmosphere, treated in the last section, there are also more local disturbances, arising from a greater rarefaction of the atmosphere over limited portions of the earth's surface, which give rise to the various irregularities in its motions, including cyclones or revolving storms, tornadoes, and water-spouts. When, on account of greater heat, or a greater amount of aqueous vapor, the atmosphere at any place becomes more rare than the surrounding portions, it ascends, and the surrounding heavier atmosphere flows in below, to supply its place, while a counter current is consequently produced above. As the lower strata of atmosphere generally contain a certain quantity of aqueous vapor, which is condensed after arising to a certain height, and forms clouds and rain, the caloric given out in the condensation, in accordance with ESPY's theory, produces a still greater rarefaction, and doubtless adds very much to the disturbance of equilibrium, and to the motive power of storms. So long, then, as the ascending atmosphere over the area of greater rarefaction is supplied with aqueous vapor by the current flowing in from all sides below, the disturbance of equilibrium must continue, and consequently the local disturbances of the atmosphere to which it gives rise, whether those of an ordinary rain storm, or a cyclone, may continue many days, while the general motions of the atmosphere may carry this disturbed area several thousands of miles.

59. In the ordinary rain-storms of the United States, the area of greater rarefication seems to be, in general, very oblong in the direction of the meridians, as is shown by Espy's charts. The atmosphere becoming more rare over the land, a current seems to set in from the Atlantic towards the Rocky Mountains, causing an ascent of the atmosphere in the west, and a line of greatest rarefaction in the direction of the meridians, arising from the condensation of the ascending vapor into clouds and rain, while the general motion of the atmosphere eastward, in those latitudes, carries this area of greater rarefaction, with its accompanying rain-storm, towards the east, at an average rate of about 30 miles per hour. As the velocity of the general eastward motion of the atmosphere is greater above, the rainy portion of the storm is for the most part on the east side of the line of greatest rarefaction, and as the currents below must be towards this line on both sides, when it passes over any place, the rain generally ceases and the wind changes.

60. When the area of rarefaction is such as to cause the atmosphere to flow in from all sides below towards a centre, and the reverse above, the disturbed portion of atmosphere, if it were not that its motions are resisted by the earth's surface, and the surrounding undisturbed part, would assume the outline and the gyratory motion in the case of no resistance, as represented in Fig. 3 and Fig. 4. But on account of the resistances, the motions of the atmosphere are very much modified, so that it has only a tendency to assume in some measure those motions, and instead of the atmosphere's receding entirely from the centre, on account of the rapidity of the gyrations near the centre, as represented in Fig. 3, it is only a little depressed in the middle, as represented in Fig. 6.

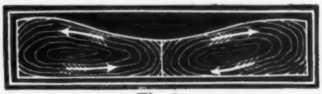

Fig 6

61. Since the force which produces the gyrations depends upon $D_t\, r$, that is, upon the velocity of the flow to and from the centre,

it is evident, that, at the centre and at the external part of the disturbed portion of atmosphere, where $D_t r$ must vanish, the resistances destroy all gyratory motion. Hence, instead of very rapid gyrations near the centre, as in the case of no resistances, there must be a calm there, and the most rapid gyrations be at some distance from the centre, in accordance with observation. The diameter of the comparatively calm portion, in the centre of the large cyclones, is sometimes about 30 miles. The velocity of gyration of the external part, which, in the case of no resistances, is small, is in a great measure destroyed by the resistances of the surrounding atmosphere, so that it is, for the most part, insensible to observation, and only the more rapid gyrations of the internal part are observed. The motion of gyration combined with the motion at the earth's surface towards the centre, gives rise to a spiral motion towards the centre, exactly in accordance with the observed motions of the atmosphere in great storms or hurricanes, as has been shown by Redfield, in a number of papers on the subject, published in the *American Journal of Science.*

62. According to (§ 29) the gyrations of the inner part of a cyclone must be from right to left in the northern hemisphere, and the contrary in the southern, which is the observed law of storms in all parts of the world, as shown by Redfield, and also by Reid, in his *Law of Storms.* It is also evident that at the equator, where $\cos \theta$ vanishes, there cannot be a cyclone, and hence, of all those which Redfield has investigated, and given in his charts of their routes, none have been traced within 10° of the equator. The typhoons or cyclones, also, of the China sea, have never been observed within 9° of the equator.

63. That the atmosphere must run into a gyration, if it converge towards a centre, is evident from the principle demonstrated in (§ 32), by which, in flowing in from all sides towards the centre,

the atmosphere must be deflected to the right in the northern hemisphere, and consequently receive a gyratory motion around that centre, from right to left, and the contrary in the southern hemisphere. Near the equator, this deflecting force vanishes, and consequently there are no cyclones there, as has been shown.

64. Since the atmosphere is depressed in the middle of cyclones, they must sensibly affect the barometer; and this is the true cause of all the great barometrical oscillations, as was first suggested by Redfield. As the cyclone approaches, there is generally a very slight rise of the barometrical column, which is at its maximum at the greatest accumulation near the external part of the cyclone, after which it is gradually depressed, until the middle of the cyclone arrives, where the atmosphere is most depressed, when the barometer is at its minimum, and then it returns in a reverse manner to its former height, when the cyclone has passed. In great storms the mercury sometimes falls more than two inches. In oblong storms, and all imperfectly developed cyclones, the same phenomena must take place in some measure, as in a complete cyclone. We have reason to conclude, therefore, that nearly all the oscillations of the barometer are caused by a cyclonic motion of the atmosphere, by which it is depressed in the middle of the cyclones. The cyclones may be very irregular and imperfectly developed, and not of sufficient violence to produce a strong wind, and several may frequently interfere with one another, so that the oscillations may frequently be very slight ones only, and very irregular.

Since the gyratory motion of a cyclone, and the consequent depression at the centre, depend upon a term containing as a factor, $\cos\theta$, (§ 29), which is the sine of the latitude, according to the preceding theory of barometrical oscillation, the oscillations should be small near the equator, and increase towards the poles, somewhat as the sine of the latitude. Accordingly, at the equator, the mean

monthly range of oscillation is only two millimetres, or less than $\frac{1}{10}$ of an inch, while there is a gradual increase with the latitude; so that at Paris it is 23.66^{mm}, and at Iceland, 35.91^{mm}. (KAEMTZ's *Meteorology, by* C. WALKER, page 297.)

65. The greater rarefaction of the atmosphere at some times than at others, without doubt, has considerable effect upon the barometer; but the theory which attributes the whole of the barometrical oscillations to the rarefaction of the atmosphere produced by the condensation of vapor in the formation of clouds and rain, cannot be maintained; for according to that theory, in the rainy belt near the equator, where there are always copious rains during the day, which are succeeded by a clear atmosphere during the night, the oscillations of the barometer should be greatest, and towards the poles, where there is little condensation of vapor into rain, they should be the least; but we have seen that just the reverse of this is true.

66. When the disturbance of equilibrium is great, but extends over a small area only, the centripetal force is much greater than in the case of large cyclones, and the gyrations are then very rapid and very near the centre, as in the case of tornadoes. Tornadoes generally occur when the surface of the earth is very warm, and the atmosphere calm. For then the strata near the surface become very much rarefied, and are consequently in a kind of unstable equilibrium for a while, when from some slight cause, the rarefied atmosphere rushes up at some point through the strata above, and consequently flows in rapidly from all sides below, and then, unless the sum of all the initial moments of gyration around the centre is exactly equal 0, which can rarely ever be the case, it must run into rapid gyrations near the centre, and a tornado is the consequence. This may be exemplified by the flowing of water through a hole in the bottom of a vessel. If the fluid at the beginning is entirely at rest, it runs out without any gyrations; but if there is the least per-

ceptible initial gyratory motion, it runs into very rapid gyrations near the centre.

67. In the case of tornadoes, which are always of small extent, the influence of the earth's rotation in producing gyratory motions is generally very small in comparison with that of the initial state of the atmosphere, as may be seen by examining equation (42). For if the atmosphere have a very small initial gyratory motion, the term u' depending upon the initial state, will be large in comparison with $n \cos \varphi$ depending upon the earth's rotation, and hence the value of $D_t \mu$, the angular velocity of gyration, depends principally upon the initial gyratory state of the atmosphere with regard to the centre of the tornado, and may be either from right to left, or the contrary. Hence there may be tornadoes at the equator, although here cannot be large cyclones. In large cyclones, the effect of the initial state, except at the equator, is insignificant in comparison with the influence of the earth's rotation; and the latter, moreover, is a constant influence, while the former is soon destroyed by resistances. Hence large cyclones are of long duration, while small tornadoes, depending principally upon the initial gyratory state for their violence, are soon overcome by the resistances.

68. On account of the centrifugal force arising from the rapid gyrations near the centre of a tornado, it must frequently be nearly a vacuum. Hence, when a tornado passes over a building, the external pressure, in a great measure, is suddenly removed, when the atmosphere within, not being able to escape at once, exerts a pressure upon the interior of perhaps nearly fifteen pounds to the square inch, which causes the parts to be thrown in every direction to a great distance. For the same reason, also, the corks fly from empty bottles, and every thing with air confined within, explodes.

69. When a tornado happens at sea, it generally produces a

water-spout. This is generally first formed above, in the form of a cloud, shaped like a funnel or inverted cone. As there is less resistance to the motions in the upper strata than near the earth's surface, the rapid gyratory motion commences there first, when the upper strata of the agitated portion of atmosphere have a tendency to assume somewhat the form of the strata in the case of no resistance, as represented in Fig. 3. This draws down the strata of cold air above, which, coming in contact with the warm and moist atmosphere ascending in the middle of the tornado, condenses the vapor and forms the funnel-shaped cloud. As the gyratory motion becomes more violent, it gradually overcomes the resistances nearer the surface of the sea, and the vertex of the funnel-shaped cloud gradually descends lower, and the imperfect vacuum of the centre of the tornado reaches the sea, up which the water has a tendency to ascend to a certain height, and thence the rapidly ascending spiral motion of the atmosphere carries the spray upward, until it joins the cloud above, when the water-spout is complete. The upper part of a water-spout is frequently formed in tornadoes on land.

When tornadoes happen on sandy plains, instead of water-spouts they produce the moving pillars of sand which are often seen on sandy deserts.

70. The routes of cyclones in all parts of the world, which have been traced throughout their whole extent, have been found to be somewhat of the form of a parabola, as represented in Fig. 7. Commencing generally near the equator, the cyclone at first

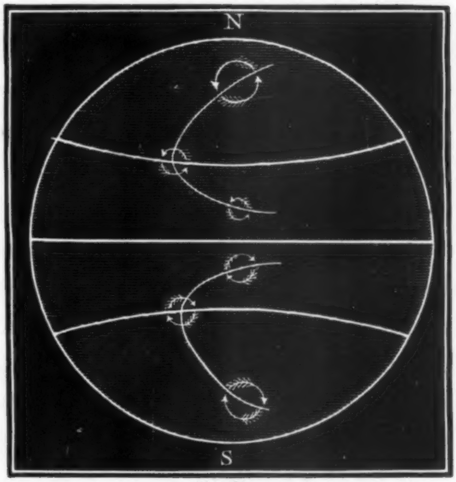

Fig. 7

moves in a direction only a little north or south of west, according to the hemisphere, when its route is gradually recurvated towards the east, having its vertex in the latitude of the tropical calm belt, as represented in the figure. This motion of a cyclone may be accounted for by means of what has been demonstrated in (§ 31), which is, that if any body, whether fluid or solid, gyrates from right to left, it has a tendency to move toward the north, but if from left to right, towards the south. Hence the interior and most violent portion of a cyclone, always gyrating from right to left in the northern hemisphere, and the contrary in the southern, must always gradually move towards the pole of the hemisphere in which it is. While between the equator and the tropical calm belt, it is carried westward by the general westward motion of the atmosphere there, but after passing the tropical calm belt, the general motion of the atmosphere carries it eastward, and hence the parabolic form of its route is the resultant of the general motions of the atmosphere, and of its gradual motion toward the pole.

It may be seen from equation (52), that the tendency of a gyrating mass to move towards the pole is as $\sin \psi$, or the cosine of the latitude, and the square of the diameter of the gyrating mass. Hence, near the equator, where the dimensions of the cyclone are always small, it moves slowly toward the pole, but as it gradually increases its dimensions, after passing its vertex, its motion towards the pole, and also its eastward motion, are both increased, and hence its progressive motion in its route or orbit is then accelerated, in accordance with the observations of REDFIELD.

71. By comparing equations (27) and (44), it is seen that they are very similar, and consequently the motions which satisfy them must be also similar. Hence the general motions of the atmosphere are similar to those of a cyclone. For the general motions of the atmosphere in each hemisphere, form a grand cyclone having the

pole for its centre, and the equatorial calm belt for its limit. But the denser portion of the atmosphere in this case being in the middle instead of the more rare, instead of ascending it descends at the pole or centre of the cyclone.

The southern cyclone having the more rapid motions on account of the resistances from the earth's surface being less, causes a greater depression of the atmosphere there than in the northern cyclone, and throws the calm belt a little north of the equator, as has been explained.

The tendency of the smaller local cyclones, as has been seen, is to run into the centres of the grand hemispherical cyclones, and thus to be swallowed up and become a part of them.

[To be continued.]

THE ELEMENTS OF QUATERNIONS.

By W. P. G. BARTLETT, Cambridge, Mass.

[Continued from Page 31.]

III. — TENSORS AND VERSORS.

10. If α and β have the same direction, the quaternion $q = \beta \div \alpha$ degenerates into a real and positive number, expressing the numerical ratio of the length of β to that of α, and is then called a *Tensor*. If, in this case, $\alpha = 1$, the tensor $q = \beta \div 1$ expresses the length of the line β, and is called the *tensor of the line* β. The tensor of β is written $T\beta$. The algebraic sum of two or more tensors is evidently a tensor; and, by § 7, a tensor may be applied to any line in space without regard to its direction. Tensors, then, satisfy the condition of § 9, and are commutative in combination with any quaternion.

11. If $T\alpha = T\beta$, the quaternion $q = \beta \div \alpha$ degenerates into the single operation of turning the line α around some axis till it

coincides with β, and is then called a *Versor*. An axis perpendicular to both α and β, and such that rotation around it from the positive direction of α to that of β is positive,* is called the *Axis* of the versor. The angle $\frac{\beta}{\alpha}$ measured positively is called the *Angle* of the versor; and is equal, as usual, to the angle $\left(-\frac{\alpha}{\beta}\right)$.

12. The lines α and β being given, let γ be a line having the same direction as α, and let $\mathrm{T}\gamma = \mathrm{T}\beta$. These conditions completely determine γ. Then, by § 9, as α, β, and γ are co-planar,

$$q = \beta \div \alpha = (\beta \div \gamma)(\gamma \div \alpha) = (\gamma \div \alpha)(\beta \div \gamma);$$

but $\gamma \div \alpha$ is a tensor, and $\beta \div \gamma$ is a versor. Any given quaternion, then, may be resolved into a product of two other determinate quaternions, one of which is a tensor and the other a versor; in this case the former is called the *tensor of the given quaternion*, and the latter *its versor*. The tensor of a quaternion, q, is written $\mathrm{T}q$, and its versor, $\mathrm{U}q$; thus

$$(1) \qquad q = \mathrm{T}q\,.\,\mathrm{U}q = \mathrm{U}q\,.\,\mathrm{T}q.$$

The axis and the angle of $\mathrm{U}q$ may be written $\mathrm{Ax.}\,\mathrm{U}q$ and $\angle\,\mathrm{U}q$; or simply $\mathrm{Ax.}\,q$ and $\angle\,q$, and called the axis and the angle of q. If the axes of three or more quaternions are co-planar, it follows that their planes intersect in a common line. If $\beta = \alpha$, then $q = \alpha \div \alpha = 1$, and $\mathrm{T}q = 1$, $\mathrm{U}q = 1$.

13. If two or more quaternions are equal, their tensors must, by § 12, be equal, and also their versors. As tensors are commutative, we have

$$\Pi q = \mathrm{T}\,\Pi q\,.\,\mathrm{U}\,\Pi q = \Pi\mathrm{T}q\,.\,\Pi\mathrm{U}q;$$

but evidently

$$(2) \qquad \mathrm{T}\,\Pi = \Pi\mathrm{T}, \qquad \text{whence also} \qquad \mathrm{U}\,\Pi = \Pi\mathrm{U}.$$

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* Either direction of rotation may be arbitrarily assumed as the positive one.

We see, however, from § 8, that we do *not* have in general $T\Sigma = \Sigma T$; and since $U\Sigma q$ must depend on the values of the tensors of the quaternions under the sign Σ, while $\Sigma U q$ does not depend on these values, it is also evident that we do *not* have in general $U\Sigma = \Sigma U$.

14. To determine Ax. q, that is, to fix the direction of a line in space, requires two independent elements (such as latitude and longitude, altitude and azimuth, &c.). A quaternion, therefore, involves *four* independent elements, — two to fix its axis, and two more for its angle and tensor; and from this fact its name, *quaternion*, is derived. If two quaternions have their four elements equal each to each, these quaternions must be equal.

15. If in the equations of § 9 we make $\gamma = -\beta$, we have $p = -1$, and $pq = -q$; whence

(3) $$T(-q) = Tq, \quad \text{Ax.}(-q) = \text{Ax.}q, \quad \angle(-q) = 180° + \angle q.$$

If now we consider one axis the negative of another, when it has the opposite direction, a positive rotation around Ax. q is equivalent to an equal amount of negative rotation around $-$Ax. q; therefore instead of (3) we may have

(3′) $$T(-q) = Tq, \quad \text{Ax.}(-q) = -\text{Ax.}q, \quad \angle(-q) = 180° - \angle q.$$

(4) 16. If $$Tp = Tq, \quad \text{Ax.}p = -\text{Ax.}q, \quad \angle p = \angle q,$$

or, what is the same thing, if

(4′) $$Tp = Tq, \quad \text{Ax.}p = \text{Ax.}q, \quad \angle p = -\angle q,$$

then p is called the *Conjugate* of q, and is written $p = Kq$.

If $p = Kq$, evidently

(5) $$q = Kp = KKq = K^2 q, \qquad q \, . \, Kq = Kq \, . \, q = (Tq)^2;$$

and by substituting Kq for q in (1) we have, since, by (4), $UK = KU$,

(6) $$Kq = TKq \, . \, UKq = Tq \, . \, KUq;$$

and by making the same substitution in (3)

$$\mathrm{T}(-\mathrm{K}q) = \mathrm{TK}q = \mathrm{T}q, \quad \mathrm{Ax.}(-\mathrm{K}q) = \mathrm{Ax.K}q = -\mathrm{Ax.}q,$$
$$\angle(-\mathrm{K}q) = 180^\circ + \angle\mathrm{K}q = 180^\circ + \angle q;$$

but $\mathrm{TK}(-q) = \mathrm{T}(-q) = \mathrm{T}q$, $\mathrm{Ax.K}(-q) = -\mathrm{Ax.}(-q) = -\mathrm{Ax.}q$,
$$\angle\mathrm{K}(-q) = \angle(-q) = 180^\circ + \angle q;$$

(7) whence, by § 14, $\quad \mathrm{K}(-q) = -\mathrm{K}q.$

17. Let $\mathrm{T}\alpha = \mathrm{T}\beta = \mathrm{T}\gamma$, and denote the versors, $\beta \div \alpha$, $\gamma \div \beta$, $\gamma \div \alpha$, respectively by u, u', u''. Then if

$$(\gamma \div \beta)(\beta \div \alpha) = \gamma \div \alpha = u'u = u'',$$

also $\quad (\alpha \div \beta)(\beta \div \gamma) = \alpha \div \gamma = \mathrm{K}u\,.\,\mathrm{K}u' = \mathrm{K}u'';$

whence, by (6),

(8) $\mathrm{K}pq^{*} = \mathrm{T}pq\,.\,\mathrm{KU}pq = \mathrm{T}q\,.\,\mathrm{T}p\,.\,\mathrm{KU}q\,.\,\mathrm{KU}p = \mathrm{K}q\,.\,\mathrm{K}p.$

As taking the conjugate of a quaternion amounts merely to changing the arbitrarily assumed positive direction of rotation to its opposite, we have

(9) $$\mathrm{K}\Sigma = \Sigma\mathrm{K}.$$

18. If any number of quaternions have a common axis, this axis will evidently be the axis of their product, and the angle of their product will be the algebraic sum of their angles. If all these factors are equal their product will be a *power* of a quaternion. Negative and fractional powers are defined by the equations

(10) $\quad q^{-m}q^m = 1 \quad$ and $\quad q^{\frac{m}{n}}q^n = q^m,$

or $\quad q^{-m} = 1 \div q^m \quad$ and $\quad q^{\frac{m}{n}} = q^m \div q^n,$

in which m and n are real integer numbers. Evidently, for all real values of m, integral or fractional,

* The conjugate of pq is here written $\mathrm{K}pq$. Hamilton uses $\mathrm{K}\,.\,pq$. The product of q multiplied by $\mathrm{K}p$ is uniformly written $\mathrm{K}p\,.\,q$. The same distinction is made with all symbols employed.

(11) $\mathrm{T}\,q^{m}$* $= (\mathrm{T}\,q)^m$, $\mathrm{U}\,q^m = (\mathrm{U}\,q)^m$, $\mathrm{Ax}.\,q^m = \mathrm{Ax}.\,q$, $\angle q^m = m \angle q$.†
After substituting q^m for q in (4), and $\mathrm{K}\,q$ for q in (11), a comparison of these equations leads, by a similar process to that used in obtaining (7), to the equation

(12) $$\mathrm{K}\,q^m = (\mathrm{K}\,q)^m.$$

The first of equations (10) is equivalent, by §§ 12 and 13, to the two equations

(10′) $$\mathrm{T}\,q^{-m} \,.\, \mathrm{T}\,q^m = 1, \quad \text{and} \quad \mathrm{U}\,q^{-m} \,.\, \mathrm{U}\,q^m = 1;$$

whence, since, by (4), $\mathrm{U}\,q^m \,.\, \mathrm{U}\,\mathrm{K}\,q^m = 1$, it follows that

(13) $$\mathrm{U}\,q^{-m} = \mathrm{U}\,\mathrm{K}\,q^m;$$ whence also, by (10′),

(14) $$q^{-m} = \mathrm{T}\,q^{-m} \,.\, \mathrm{U}\,q^{-m} = \mathrm{T}\,q^{-m} \,.\, \mathrm{U}\,\mathrm{K}\,q^m = \mathrm{T}\,q^{-2m} \,.\, \mathrm{K}\,q^m.$$

By putting $p\,q$ for q and $m = 1$ in (14), we get by (8) and (14),

(15) $$(p\,q)^{-1} = (\mathrm{T}\,p\,q)^{-2}\,[\mathrm{K}\,p\,q = \mathrm{T}\,q^2 \,.\, q^{-1} \,.\, \mathrm{T}\,p^2 \,.\, p^{-1}] = q^{-1} \,.\, p^{-1}.‡$$

---

* The tensor of q^m is here written $\mathrm{T}\,q^m$, and the m^{th} power of $\mathrm{T}\,q$, $(\mathrm{T}\,q)^m$. HAMILTON uses $\mathrm{T}\,.\,q^m$ and $\mathrm{T}\,q^m$ respectively. The same distinction is made with all symbols employed.

† It should be observed, that, when m is fractional, since $\angle q = t\,360° + \angle q$, $\angle q^m = m\,(\angle q + t\,360°)$; and in such cases $\angle q^m$ will have a certain number of different values for different integral values of t. For convenience, however, we suppose $t = 0$, when nothing is said to the contrary, so that $\angle q^m = m \angle q$ in all cases.

‡ The geometry of this and some previous propositions may be shown as follows, *supposing p and q to be only versors.* Let a sphere be described with a radius of unity from the common origin of the lines α, &c. as a centre. Let the intersections of this sphere with these lines be connected by great circle arcs. Then, by §§ 5, 7, and 9, we may represent q by either of the equal arcs, $\alpha\,\beta$, or $\beta\,\delta$, and p by $\varepsilon\,\beta$, or $\beta\,\gamma$. Then $p\,q$ is represented by $\alpha\,\gamma$, and $q\,p$ by $\varepsilon\,\delta$. Equations (4′) and (11) show that $\mathrm{K}\,q$ and q^{-1} are equal, *when q is a versor*, and that either may be represented by $\beta\,\alpha$ or $\delta\,\beta$. A simple examination of the accompanying figure shows the meaning of (15) and §§ 7 and 9, *as far as versors are concerned;* and as tensors are mere numerical elements following the ordinary rules of arithmetic, they add no difficulty to these cases.

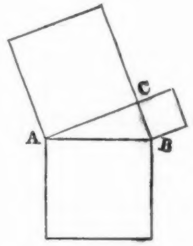
A SECOND BOOK IN GEOMETRY.

[Continued from Page 410, Vol. I.]

CHAPTER VI.

THE PYTHAGOREAN PROPOSITION.

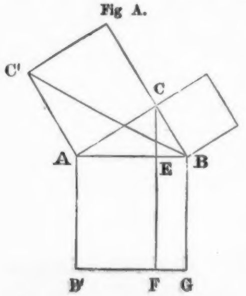
64. We recollect that the square built on the hypotenuse of a right triangle is equivalent in its area to the sum of the squares built upon its legs. This is one of the most useful of all geometrical truths. Let us first analyze it in one or two modes, and then build it up synthetically by the same paths. We may afterwards, if we like, devise other modes of analysis and synthesis, for this proposition, like all others, may be approached in various ways.



65. The Pythagorean proposition or theorem might be suggested in different ways. But in whatever way we were led to suspect that the square on the hypotenuse is equivalent to the sum of the squares on the legs, we should, in reflecting upon it, probably begin by drawing a right triangle with a square built upon each side.

66. We should inquire whether the square on the hypotenuse could be divided into two parts that should be respectively equal to the other two squares. And we should judge that these parts should be somewhat similar to each other in shape, because the legs do not differ in their relations to the hypotenuse except in size, and in the angles they make with it.

Fig A.



67. But we cannot readily conceive of any division of the square into two somewhat similar parts, except into two rectangles. And then it is apparent that two rectangles bearing respectively the same relations to the squares on the legs, may be found by drawing a line from the vertex of the right angle at right angles with the hypotenuse, and continuing it through the square, as C F is here drawn.

68. It will now only be necessary to show that one of these rectangles is equivalent to its corresponding square; because the same mode of proof will obviously answer for the other rectangle and its square.

69. Now if we know, or can prove, that the area of a rectangle is measured by the product of its sides, we shall have to prove that $AE \times AB'$ or $AE \times AB$ is equivalent to $AC \times AC'$.

70. But by the doctrine of proportion it may be shown that this would be equivalent to saying that A E is to A C as A C is to A B.

71. Again, it may be shown by geometry that this proportion between the lines A B, A C, and A E, would be true if the triangle A E C were similar to A C B, and that A E stood in one to A C, as A C stood to A B in the other; so that all that remains for us to do is to show that these triangles are similar.

72. But we can show by geometry that two triangles are similar when their angles are equal.

73. And it is easy to show that the angles of these triangles are equal to each other.

74. For C A B and C A E are the same angle; A C B and A E C are both right angles, and therefore A B C and A C E are each complements of C A E. Moreover, A C and A E are situated in the triangle A E C, in the same manner that A B and A C are situated in the triangle A B C.

75. We have thus, in articles 66 - 74, sufficiently analyzed the Pythagorean proposition to enable us to build it up again in a deductive form. This analysis, however, has been partly algebraical, as it has introduced the idea of multiplying two lines to produce a surface. Let us now begin and build up the proposition by the same road. We shall find 31 articles necessary, and I will number them from 76 to 106.

First Proof of the Pythagorean Proposition.

76. *Definition.* The comparative size of two quantities is called their ratio; thus if one is twice as large as the other, they are said to be in the same ratio as that of 2 to 1; or to be in the ratio 2 to 1; or it is said, in a looser way, that their ratio equals 2.

77. *Notation.* Ratio is written by means of the marks :, $\div$, and by writing one quantity over the other. Thus, $A : B$, $A \div B$, and $\frac{A}{B}$, are each used to signify the ratio of A to B. These marks are the same as those used in arithmetic to signify Quotient, because the meaning of a quotient is "a number having the same ratio to one, that the dividend has to the divisor." The ratio of A to B is not the quotient of A divided by B, but it is the ratio of that quotient to unity.

78. *Axiom.* If each of two quantities is multiplied or divided by the same number, the ratio of the products or quotients will be the same as that of the quantities themselves. Thus twenty inches is in the same ratio to twenty rods as one inch to one rod, or as the twentieth of an inch to the twentieth of a rod.

79. *Definition.* A proportion is the equality of two ratios. Thus (if we use the sign $=$ to signify "is equal to") $A : B = C : D$ is the statement of a proportion. It signifies that A is in the same proportion to B that C is to D.

80. *Definition.* When a proportion is written as in article 79, the first and last terms, that is, A and D, are called the extremes, and the others, that is, B and C, are called the means.

81. *Theorem.* In every proportion the product of the means is equal to that of the extremes. —*Proof.* In any proportion, as $M : N = P : Q$, we wish to prove (using the mark $\times$ to signify "multiplied by") that $M \times Q = N \times P$. Now in order to do this, we must use only self-evident truths. The only truth of this character that we have given above is that of article 78. But in order, by means of the multiplications of article 78, to change the first ratio $M : N$ into $M \times Q$, we must, whatever else we do, at least multiply each term by Q, and this will give us $M \times Q : N \times Q = P : Q$, and in order to change the second ratio $P : Q$ into $N \times P$, we must, at all events, multiply each term by N, and this will give us $M \times Q : N \times Q = N \times P : N \times Q$.

Thus from the self-evident truth of article 78 we find that the product of the means bears the same ratio to the product $N \times Q$ that is borne to it by the product of the extremes. And as it is self-evident that two quantities, bearing the same ratio to a third, must be equal to each other, we have proved that the product of the means is equal to that of the extremes.

82. *Definition.* When both the means are the same quantity, that quantity is called a mean proportional between the extremes.

83. *Corollary.* It follows from article 81, that the product of the mean proportional multiplied by itself is equal to the product of the extremes.

84. *Definitions.* A *unit* of length is a line taken as a standard of comparison for lengths. Thus an inch, a foot, a pace, a span, etc., are units. The *length* of any line is its ratio to the unit of length.

85. *Definition.* A unit of surface is a surface taken as a standard of comparison. The most common unit of surface is a square whose side is a unit of length.

86. *Definition.* The area of a surface is the ratio of the surface to the unit of surface.

87. *Theorem.* Any straight line in the same plane with two parallel lines makes the same angle with one that it does with the other. — *Proof.* For as the straight line has but one direction, and each of the parallel lines may always be considered as going in the same direction as the other, the difference of that direction from the direction of the third straight line must be the same for each of the parallel lines.

88. *Corollary.* If a straight line is parallel to one of two parallel lines, it is parallel to the other; if at right angles to one of the two, it is at right angles to the other.

89. *Theorem.* If a straight line make on the same side of itself the same angle with two other straight lines in the same plane, those other straight lines must be parallel.

Scholium. The line must not be conceived as reversing its direction at any point. — *Proof.* For if two directions differ equally from a third, they must be equal to each other.

90. *Axiom.* If the boundaries of one plane surface are similar to those of another in such a way that the two surfaces would coincide in extent if laid one upon the other, the two surfaces are equivalent.

[To be continued.]

Editorial Items.

R. T. Massie, Esq., late Assistant Instructor in Mathematics in the University of Virginia, has accepted the Professorship of Mathematics in Randolph-Macon College, Va.

Daniel Vaughan, Esq., has accepted the Professorship of Mathematics in Masonic College, at Lagrange, Ky.

Prof. William H. Young has been transferred from the Professorship of Mathematics to that of Ancient Languages in the Ohio University at Athens. Eli T. Tappan, of Steubenville, Ohio, has been appointed to the Professorship of Mathematics thus vacated.

Prof. J. W. Patterson, of Dartmouth College, has been transferred from the Professorship of Mathematics to that of Astronomy, made vacant by the decease of Prof. Ira Young.

All solutions and communications for the Monthly should be addressed to the Editor, at Cambridge, Mass., and all remittances and business letters should be sent to Messrs. Ivison and Phinney, 48 and 50 Walker St., New York. A compliance with this request will save us much trouble.

Erratum. — In Dem. 19, p. 49, for B T U K read B T U H.

Books Received. — Place of Mathematics in University Education. Inaugural Address of Charlton S. Lewis, Professor of Pure Mathematics in Troy University, delivered before the Trustees at their annual meeting, July 20th, 1859. — Nouvelles Annales de Mathématiques. Octobre, 1859. — A Descriptive Catalogue of Text-Books for Schools and Colleges. Published by Ivison and Phinney, 48 and 50 Walker St., New York. Edition of September, 1859, 8vo. pp. 168. Gratis to teachers and those especially interested in Education. This volume is filled with recommendations of the Publishers' series of books. They express the deliberate opinions of educated men in all parts of the country. Teachers will do well to procure a copy of this Catalogue for reference; as it contains the idea of a large number of practical teachers as to what constitutes a good Text-Book.

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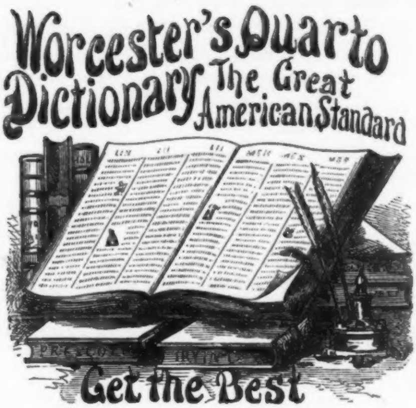

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